An Alternative Basis for the Wigner-Racah Algebra of the Group SU(2)

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Abstract

The Lie algebra of the classical group SU(2) is constructed from two quon algebras for which the deformation parameter is a common root of unity. This construction leads to (i) a (not very well-known) polar decomposition of the generators J_- and J_+ of the SU(2) Lie algebra and to (ii) an alternative to the $\{J^2, J_3\}$ quantization scheme, viz., the $\{J^2, U_r\}$ quantization scheme. The key ideas for developing the Wigner-Racah algebra of the group SU(2) in the $\{J^2, U_r\}$ scheme are given. In particular, some properties of the coupling and recoupling coefficients as well as the Wigner-Eckart theorem in the $\{J^2, U_r\}$ scheme are briefly discussed.

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1 Motivations and Introduction

In recent years, intermediate statistics and deformed statistics were the object of considerable interest [1-19]. The use of deformed oscillator algebras proved to be useful in parastatistics, anyonic statistics and deformed statistics. In particular, one- and two-parameter deformations of the Bose-Einstein statistics (more precisely, deformations of the relevant second quantization formalism) were studied by several authors [6-19]. A common characteristics of most of these studies is that it is possible to obtain a Bose-Einstein condensation of a free gas of bosons in $D = 2$ and 3 dimensions. However, in $D = 3$ dimensions, the q-deformed Bose-Einstein (B-E) temperature is generally greater than the classical (corresponding to $q = 1$) B-E temperature. In the specific case of ^{4}He super-fluid in phase II, the usual q-deformations, i.e., the *à la* Biedenharn [20] and *à la* Macfarlane [21] q-deformations, yield the following inequality :

$$(T_{\mathrm{B-E}})_{q\neq 1} > (T_{\mathrm{B-E}})_{q=1} > (T_{\mathrm{B-E}})_{\mathrm{exp}}$$

so that we do not gain anything when passing from $q = 1$ to $q \neq 1$. On the other hand, by using a *à la* Rideau [22,23] deformation, it is feasible to lower the critical temperature $(T_{\mathrm{B-E}})_{q\neq 1}$ due to the occurrence of a second parameter ν_0' in addition to the deformation parameter q. This result corresponds to the model M_1 introduced in ref.[19]. For this model, we can obtain couples (ν_0', q) for which $(T_{\mathrm{B-E}})_{q\neq 1}$ is in agreement with the experimental value $(T_{\mathrm{B-E}})_{\mathrm{exp}} \sim 2.17$ K. However, as a drawback, the model M_1 depends on two parameters. Although it is possible to find a physical interpretation (in terms of the chemical potential) of the deformation parameter q, there is up to now no satisfying interpretation of the phenomenological parameter ν_0'.

The just mentioned difficulty to interpret the parameter ν_0' was the starting point of an investigation of alternative deformations of the second quantization formalism. More specifically, we investigated the *à la* Arik and Coon [24] deformation but in the case where q is a root of unity. (In the original work by Arik and Coon, the deformation parameter q is a real number : The reality of q ensures that the creation and annihilation operators are connected via Hermitean conjugation.) Thus, we arrived at the conclusion that it is necessary to simultaneously consider two quon algebras A_q and $A_{\bar{q}}$ in order to obtain a convenient framework for obtaining B-E condensation of quons.

As a first by-product, we were naturally left to the definition and study of operators, referred to as k-fermion operators, that interpolate between boson and fermion operators. These new operators arise through the consideration of two non-commuting quon algebras A_q and $A_{\bar{q}}$ for which $q = \exp(2\pi\mathrm{i}/k)$ with $k \in \mathbf{N} \setminus \{0, 1\}$. The case $k = 2$ corresponds to fermions and the limiting case $k \to \infty$ to bosons. Generalized coherent states (connected to k-fermionic states) and super-coherent states (involving a k-fermionic sector and a purely bosonic sector) were examined. In addition, the operators in the k-fermionic algebra were used to find realizations

of the Dirac quantum phase operator and of the W_∞ Fairlie-Fletcher-Zachos algebra [25]. All these matters were discussed in Bregenz (at the Symposium *Symmetries in Science X*), Dubna (at the VIII International Conference on *Symmetry Methods in Physics*) and Istanbul (at the International Workshop *Quantum Groups, Deformations and Contractions*) and shall be reported elsewhere [26,27].

In the present paper, we would like to deal with a second by-product of our quon approach. Here, instead of considering two non-commuting quon algebras A_q and $A_{\bar{q}}$, we shall consider two realizations of two commuting quon algebras corresponding to the same root of unity $q = \exp(2\pi \mathrm{i}/k)$ with $k \in \mathbf{N} \setminus \{0,1\}$. We shall see how to construct (in Section 2) the Lie algebra of SU(2) from these two quon algebras ; how to obtain (in Section 3) an alternative to the $\{J^2, J_z\}$ scheme of SU(2) ; and how to develop (in Section 4) the Wigner-Racah algebra of SU(2) in this new scheme. In a last section (Section 5), we shall indicate some perspectives and briefly discuss some open problems.

2 A Quon Approach to SU(2)

We start with two commuting quon algebras $A_i = \{a_{i-}, a_{i+}, N_i\}$, with $i = 1$ and 2, for which the generators satisfy

$$a_{i-}a_{i+} - q a_{i+}a_{i-} = 1, \quad [N_i, a_{i\pm}] = \pm a_{i\pm} \tag{1}$$

where the deformation parameter

$$q = \exp\left(\frac{2\pi \mathrm{i}}{k}\right) \quad \text{with} \quad k \in \mathbf{N} \setminus \{0,1\} \tag{2}$$

(the same for A_1 and A_2) is a root of unity. As constraint relations, compatible with (1) and (2), we take the nilpotency conditions

$$(a_{i+})^k = (a_{i-})^k = 0 \quad \text{with} \quad k \in \mathbf{N} \setminus \{0,1\} \tag{3}$$

Grassmannian realizations of eqs.(1) and (3) are obtainable from ref.[26]. In this work, we take the representations of A_1 and A_2 defined by

$$a_{1+}|n_1) = |n_1+1), \quad a_{1+}|k-1) = 0$$

$$a_{1-}|n_1) = [n_1]_q\,|n_1-1), \quad a_{1-}|0) = 0$$

$$a_{2+}|n_2) = [n_2+1]_q\,|n_2+1), \quad a_{2+}|k-1) = 0$$

$$a_{2-}|n_2) = |n_2-1), \quad a_{2-}|0) = 0$$

$$N_1|n_1) = n_1|n_1), \quad N_2|n_2) = n_2|n_2)$$

on a Fock space $\mathcal{F} = \{|n_1 n_2) = |n_1) \otimes |n_2) : n_1, n_2 = 0, 1, \cdots, k-1\}$ of finite dimension ($\dim \mathcal{F} = k^2$). We use here the notation

$$[x]_q = \frac{1-q^x}{1-q} \quad \text{for} \quad x \in \mathbf{R}$$

so that $[n]_q = 1 + q + \cdots + q^{n-1}$ for $n \in N^*$.

We now define the two following linear operators

$$H = \sqrt{N_1 \left(N_2 + 1\right)}$$

and

$$U_r = \left[a_{1+} + \exp\left(\mathrm{i}\frac{\phi_r}{2}\right) \frac{(a_{1-})^{k-1}}{[k-1]_q!}\right] \left[a_{2-} + \exp\left(\mathrm{i}\frac{\phi_r}{2}\right) \frac{(a_{2+})^{k-1}}{[k-1]_q!}\right]$$

where the real parameter ϕ_r is taken in the form

$$\phi_r = \pi(k-1)r \quad \text{with} \quad r \in \mathbf{R}$$

and the q-deformed factorial is defined by

$$[n]_q! = [1]_q [2]_q \cdots [n]_q \quad \text{for} \quad n \in \mathbf{N}^* \quad \text{and} \quad [0]_q! = 1$$

The action of U_r on $\mathcal{F}$ is easily found to satisfy

$$U_r|n_1 n_2) = |n_1 + 1, n_2 - 1) \quad \text{for} \quad n_1 \neq k-1 \quad \text{and} \quad n_2 \neq 0 \tag{4}$$

and

$$U_r|k-1, 0) = \exp\left(\mathrm{i}\phi_r\right)|0, k-1) \tag{5}$$

while for H we have

$$H|n_1 n_2) = \sqrt{n_1(n_2+1)}|n_1 n_2) \tag{6}$$

By using the Schwinger trick

$$j = \frac{1}{2}\left(n_1 + n_2\right), \quad m = \frac{1}{2}\left(n_1 - n_2\right) \quad \Rightarrow \quad |n_1 n_2) = |j+m, j-m) \equiv |jm\rangle$$

we can rewrite eqs.(4) and (5) as

$$U_r|jm\rangle = [1 - \delta(m,j)]\,|j, m+1\rangle + \delta(m,j)\exp\left(\mathrm{i}\phi_r\right)|j, -j\rangle$$

Similarly, eq.(6) can be rewritten

$$H|jm\rangle = \sqrt{(j+m)(j-m+1)}|jm\rangle$$

Furthermore, we have

$$U_r^\dagger |jm\rangle = [1 - \delta(m, -j)]\, |j, m-1\rangle + \delta(m, -j) \exp\left(-\mathrm{i}\phi_r\right) |jj\rangle$$

where ${U_r}^\dagger$ stands for the adjoint of U_r. For a fixed value of k, we take

$$2j = k - 1 \quad \text{with} \quad k \in \mathbf{N} \setminus \{0, 1\}$$

We can thus have $j = \frac{1}{2}$, 1, $\frac{3}{2}$, $\cdots$. The case $j = 0$ corresponds to the limiting situation where $k \to \infty$.

It is obvious that the operator H is Hermitean and the operator U_r is unitary. The action of U_r on $\mathcal{F}$ is cyclic. As a further property of U_r, we have

$$(U_r)^{2j+1} = \exp(\mathrm{i}\phi_r)$$

that reflects the cyclical character of U_r.

Let us introduce the three operators

$$J_+ = H U_r, \quad J_- = U_r^\dagger H \tag{7}$$

and

$$J_3 = \frac{1}{2}\left(N_1 - N_2\right) \tag{8}$$

It is immediate to check that the action on the state $|jm\rangle$ of the operators defined by eqs.(7) and (8) is given by

$$J_\pm |jm\rangle = \sqrt{(j \mp m)(j \pm m + 1)}|j, m \pm 1\rangle$$

and

$$J_3 |jm\rangle = m |jm\rangle$$

Consequently, we have the commutation relations

$$[J_3, J_\pm] = \pm J_\pm, \quad [J_+, J_-] = 2J_3$$

which correspond to the Lie algebra of the group SU(2). As a result, the non-deformed Lie algebra su(2) is obtained from two q-deformed oscillator algebras.

To close this section, it is interesting to note that we can generate the infinite dimensional Lie algebra W_∞ from the generators of A_1 and A_2. Indeed, by putting

$$U = U_r, \quad V = q^{N_1 - N_2}$$

and

$$T_{(m_1, m_2)} = q^{m_1 m_2} U^{m_1} V^{m_2}$$

we can prove that

$$[T_m, T_n] = -2\,\mathrm{i}\sin\left(\frac{2\pi}{k} m \times n\right) T_{m+n} \tag{9}$$

where we use the abbreviations

$$m = (m_1, m_2)\,, \quad n = (n_1, n_2)$$

and

$$m + n = (m_1 + n_1, m_2 + n_2)\,, \quad m \times n = m_1 n_2 - m_2 n_1$$

Equation (9) shows that the operators T_ℓ span the algebra W_∞ introduced by Fairlie, Fletcher and Zachos [25]. This result parallels a similar result obtained in ref.[26] in the study of k-fermions and of the Dirac quantum phase operator.

3 A New Basis for SU(2)

At this stage, it is important to establish a link with the work by Lévy-Leblond [28]. The decomposition (7), in terms of H and U_r, coincides with the polar decomposition, described in ref.[28], of the shift operators J_+ and J_- of the Lie algebra su(2). This is easily seen by taking the matrix elements of U_r and H and by comparing these elements to the ones of the operators Υ and J_T in [28]. This yields $H \equiv J_T$; furthermore, by identifying the arbitrary phase φ of [28] to $\phi_r = 2\pi jr = \pi(k-1)r$, we obtain that U_r turns out to be identical to the operator Υ of [28]. Equation (7) constitutes an important original result of ref.[28].

It is easy to prove that the Casimir operator

$$J^2 = \frac{1}{2}\left(J_+ J_- + J_- J_+\right) + J_3^2 = H^2 + J_3^2 - J_3$$

commutes with U_r for any value of r. (Note that the commutator $[U_r, U_s]$ is different from zero for $r \neq s$.) Therefore, for fixed r, the commuting set $\{J^2, U_r\}$ provides us with an alternative to the familiar commuting set $\{J^2, J_3\}$ of angular momentum theory. The (complete) set of commuting operators $\{J^2, U_r\}$ can be easily diagonalized. This leads to the following result.

Result : The spectra of the operators U_r and J^2 are given by

$$U_r|j\alpha; r\rangle = q^{-\alpha}|j\alpha; r\rangle, \quad J^2|j\alpha; r\rangle = j(j+1)|j\alpha; r\rangle \tag{10}$$

where

$$|j\alpha; r\rangle = \frac{1}{\sqrt{2j+1}} \sum_{m=-j}^{j} q^{\alpha m}|jm\rangle \tag{11}$$

with the range of values

$$\alpha = -jr, -jr+1, \cdots, -jr+2j, \quad 2j \in \mathbf{N}$$

The parameter q in eqs.(10) and (11) is

$$q = \exp\left(\frac{2\pi \mathrm{i}}{2j+1}\right) \tag{12}$$

(cf. eq.(2) with $k = 2j+1$ for $k \in \mathbf{N} \setminus \{0, 1\}$ and $k \to \infty$ for $j = 0$).

It is important to note that in eqs.(10) and (11) the label α goes, by step of 1, from $-jr$ to $-jr + 2j$. (It is only for $r = 1$ that α goes, by step of 1, from $-j$ to j.) The inter-basis expansion coefficients

$$\langle jm | j\alpha; r\rangle = \frac{1}{\sqrt{2j+1}} q^{\alpha m}$$

(with $m = -j, -j+1, \cdots, j$ and $\alpha = -jr, -jr+1, \cdots, -jr+2j$) in eq.(11) define a unitary transformation that allows to pass from the well-known orthonormal standard basis $\{|jm\rangle : 2j \in \mathbf{N},\ m = -j, -j+1, \cdots, j\}$ of the space $\mathcal{F}$ to the orthonormal non-standard basis $B_r = \{|j\alpha; r\rangle : 2j \in \mathbf{N},\ \alpha = -jr, -jr+1, \cdots, -jr+2j\}$. Then, the expansion

$$|jm\rangle = \frac{1}{\sqrt{2j+1}} \sum_{\alpha=-jr}^{-jr+2j} q^{-\alpha m} |j\alpha; r\rangle$$

with

$$m = -j, -j+1, \cdots, j, \quad 2j \in \mathbf{N}$$

is the inverse of eq.(11).

We thus foresee that it is possible to develop the Wigner-Racah algebra (WRa) of the group SU(2) in the $\{J^2, U_r\}$ scheme. This furnishes an alternative to the WRa of SU(2) in the SU(2) $\supset$ U(1) basis corresponding to the $\{J^2, J_3\}$ scheme.

4 A New Approach to the Wigner-Racah Algebra of SU(2)

In this section, we give the basic ingredients for the WRa of SU(2) in the $\{J^2, U_r\}$ scheme. The Clebsch-Gordan coefficients (CGc's) adapted to the $\{J^2, U_r\}$ scheme are defined from the SU(2) $\supset$ U(1) CGc's adapted to the $\{J^2, J_3\}$ scheme. The adaptation to the $\{J^2, U_r\}$ scheme afforded by eq.(11) is transferred to SU(2) irreducible tensor operators. This yields the Wigner-Eckart theorem in the $\{J^2, U_r\}$ scheme.

4.1 Coupling and Recoupling Coefficients in the $\{J^2, U_r\}$ Scheme

The CGc's or coupling coefficients $(j_1 j_2 \alpha_1 \alpha_2 | j\alpha; r)$ in the $\{J^2, U_r\}$ scheme are simple linear combinations of the SU(2) $\supset$ U(1) CGc's. In fact, we have

$$(j_1 j_2 \alpha_1 \alpha_2 | j\alpha; r) = \frac{1}{\sqrt{(2j_1+1)(2j_2+1)(2j+1)}} \sum_{m_1=-j_1}^{j_1} \sum_{m_2=-j_2}^{j_2} \sum_{m=-j}^{j}$$

$$\times q^{\alpha m} q_1^{-\alpha_1 m_1} q_2^{-\alpha_2 m_2} \, (j_1 j_2 m_1 m_2 | jm)$$

where q, q_1 and q_2 are given by eq.(12) in terms of j, j_1 and j_2, respectively. The symmetry properties of the coupling coefficients $(j_1 j_2 \alpha_1 \alpha_2 | j\alpha; r)$ cannot be expressed in a simple way (except the symmetry under the interchange $j_1\alpha_1 \leftrightarrow j_2\alpha_2$). Let us introduce the f_r symbol via

$$f_r \begin{pmatrix} j_1 & j_2 & j_3 \\ \alpha_1 & \alpha_2 & \alpha_3 \end{pmatrix} = (-1)^{2j_3} \frac{1}{\sqrt{2j_1+1}} \, (j_2 j_3 \alpha_2 \alpha_3 | j_1 \alpha_1; r)^* \tag{13}$$

where the star indicates the complex conjugation. Its value is multiplied by the factor $(-1)^{j_1+j_2+j_3}$ when its two last columns are interchanged. However, the interchange of two other columns cannot be described by a simple symmetry property. Nevertheless, the f_r symbol is of central importance for the Wigner-Eckart theorem in the $\{J^2, U_r\}$ scheme (see eq.(17) below).

Following ref.[29], we define a more symmetrical symbol, namely the $\bar{f}_r$ symbol, through

$$\bar{f}_r \begin{pmatrix} j_1 & j_2 & j_3 \\ \alpha_1 & \alpha_2 & \alpha_3 \end{pmatrix} = \frac{1}{\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)}} \sum_{m_1=-j_1}^{j_1} \sum_{m_2=-j_2}^{j_2} \sum_{m_3=-j_3}^{j_3}$$

$$\times \, q_1^{-\alpha_1 m_1} q_2^{-\alpha_2 m_2} q_3^{-\alpha_3 m_3} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \tag{14}$$

where the parameters q_i are given by eq.(12) with $q \equiv q_i$ and $j \equiv j_i$ for $i = 1, 2, 3$. The $3 - jm$ symbol on the right-hand side of eq.(14) is an ordinary Wigner symbol for the group SU(2) in the SU(2)$\supset$U(1) basis. As a matter of fact, it is possible to pass from the f_r symbol to the $\bar{f}_r$ symbol and vice versa by means of a metric tensor. The $\bar{f}_r$ symbol is more symmetrical than the f_r symbol. The $\bar{f}_r$ symbol exhibits the same symmetry properties under permutations of its columns as the $3-jm$ Wigner symbol : Its value is multiplied by $(-1)^{j_1+j_2+j_3}$ under an odd permutation and does not change under an even permutation. In addition, the orthogonality properties of the highly symmetrical $\bar{f}_r$ symbol easily follow from the corresponding properties of the $3 - jm$ Wigner symbol. Thus, we have

$$\sum_{j_3\alpha_3} (2j_3+1) \bar{f}_r \begin{pmatrix} j_1 & j_2 & j_3 \\ \alpha_1 & \alpha_2 & \alpha_3 \end{pmatrix}^* \bar{f}_r \begin{pmatrix} j_1 & j_2 & j_3 \\ \alpha_1' & \alpha_2' & \alpha_3 \end{pmatrix} = \delta(\alpha_1', \alpha_1)\delta(\alpha_2', \alpha_2) \tag{15}$$

and

$$\sum_{\alpha_1\alpha_2} \bar{f}_r \begin{pmatrix} j_1 & j_2 & j_3 \\ \alpha_1 & \alpha_2 & \alpha_3 \end{pmatrix} \bar{f}_r \begin{pmatrix} j_1 & j_2 & j_3' \\ \alpha_1 & \alpha_2 & \alpha_3' \end{pmatrix}^* = \frac{1}{2j_3+1} \Delta(0|j_1 \otimes j_2 \otimes j_3) \delta(j_3', j_3) \delta(\alpha_3', \alpha_3) \tag{16}$$

where $\Delta(0|j_1 \otimes j_2 \otimes j_3) = 1$ or 0 according to as the Kronecker product $(j_1) \otimes (j_2) \otimes (j_3)$ contains or does not contain the identity irreducible representation (0) of SU(2). Observe that the real number r is the same for all the $\bar{f}_r$ symbols occurring in eqs.(15) and (16).

The values of the SU(2) CGc's in the $\{J^2, U_r\}$ scheme as well as of the f_r and $\bar{f}_r$ coefficients are not necessarily real numbers. For instance, we have the following property under complex conjugation

$$\bar{f}_r \begin{pmatrix} j_1 & j_2 & j_3 \\ \alpha_1 & \alpha_2 & \alpha_3 \end{pmatrix}^* = (-1)^{j_1+j_2+j_3} \bar{f}_r \begin{pmatrix} j_1 & j_2 & j_3 \\ \alpha_1 & \alpha_2 & \alpha_3 \end{pmatrix}$$

Hence, the value of the $\bar{f}_r$ coefficient is real if $j_1 + j_2 + j_3$ is even and pure imaginary if $j_1 + j_2 + j_3$ is odd. Then, the behavior of the $\bar{f}_r$ symbol under complex conjugation is completely different as the one of the ordinary $3 - jm$ Wigner symbol.

Finally, it is worth to mention that the recoupling coefficients of the group SU(2) can be expressed in terms of coupling coefficients of SU(2) in the $\{J^2, U_r\}$ scheme. For example, the $9 - j$ symbol can be expressed in terms $\bar{f}_r$ symbols by replacing, in its decomposition in terms of $3 - jm$ symbols, the $3 - jm$ symbols by $\bar{f}_r$ symbols. On the other hand, the decomposition of the $6 - j$ symbol in terms of $\bar{f}_r$ symbols requires the introduction of six metric tensors corresponding to the six arguments of the $6 - j$ symbol. These matters may be developed by following the approach initiated in refs.[29-32].

4.2 Wigner-Eckart Theorem in the $\{J^2, U_r\}$ Scheme

From the spherical components $T_q^{(k)}$ (with $q = -k, -k+1, \cdots, k$) of an SU(2) irreducible tensor operator $\mathbf{T}^{(k)}$, we define the $2k+1$ components

$$T_\alpha^{(k)}(r) = \frac{1}{\sqrt{2k+1}} \sum_{m=-k}^{k} q^{\alpha m} T_m^{(k)}$$

with

$$\alpha = -kr, -kr+1, \cdots, -kr+2k, \quad 2k \in \mathbf{N}$$

In the $\{J^2, U_r\}$ scheme, the Wigner-Eckart theorem reads

$$\langle \tau_1 j_1 \alpha_1; r | T_\alpha^{(k)}(r) | \tau_2 j_2 \alpha_2; r \rangle = \left(\tau_1 j_1 || T^{(k)} || \tau_2 j_2 \right) f_r \begin{pmatrix} j_1 & j_2 & k \\ \alpha_1 & \alpha_2 & \alpha \end{pmatrix} \tag{17}$$

where $\left(\tau_1 j_1 || T^{(k)} || \tau_2 j_2 \right)$ denotes an ordinary reduced matrix element. Such an element is basis-independent. Therefore, it does not depend on the labels α_1, α_2 and α. On the contrary, the f_r coefficient in eq.(17), defined by eq.(13), depends on the labels α_1, α_2 and α.

5 Concluding Remarks

In this paper, we have developed a quon approach to the Lie algebra of the classical (not quantum!) group SU(2). Such an approach leads to the polar decomposition of the generators J_+ and J_- of SU(2), a decomposition originally introduced by Lévy-Leblond [28].

The familiar $\{J^2, J_3\}$ quantization scheme with the (usual) standard spherical basis $\{|jm\rangle : 2j \in \mathbf{N},\ m = -j, -j+1, \cdots, j\}$, corresponding to the canonical chain of groups SU(2)⊃U(1), is thus replaced by the $\{J^2, U_r\}$ quantization scheme with a (new) basis, namely, the non-standard basis $B_r = \{|j\alpha; r\rangle : 2j \in \mathbf{N},\ \alpha = -jr, -jr+1, \cdots, -jr+2j\}$. We have given the premises of the construction of the Wigner-Racah algebra of the group SU(2) in the B_r basis. Of course, there exists an infinity of B_r bases due to the fact that $r \in \mathbf{R}$. The case $r = 1$ probably deserves a special attention. We shall give elsewhere a complete development of the Wigner-Racah algebra of SU(2) in the B_1 basis. In particular, the calculation and the properties, including Regge symmetry properties, of the coupling coefficients ($\bar{f}_1$ and f_1 symbols and CGc's in the $\{J^2, U_1\}$ scheme) shall be the object of a forthcoming paper.

As a further interesting step, it would be interesting to find realizations of the B_r basis (i) on the sphere S^2 for j integer and (ii) on the Fock-Bargmann spaces (of entire analytical functions) in 1 and 2 dimensions for j integer or half of an odd integer. In this respect, the problem of finding a differential realization of the operator U_r on S^2 and of expressing its eigenfunctions

$$[y_r]_{\ell\alpha}(\theta, \varphi) = \frac{1}{\sqrt{2\ell+1}} \sum_{m=-\ell}^{\ell} q^{\alpha m} Y_{\ell m}(\theta, \varphi) \qquad (18)$$

with

$$\alpha = -\ell r, -\ell r+1, \cdots, -\ell r+2\ell, \quad \ell \in \mathbf{N}$$

as special functions is very appealing. (In eq.(18), $Y_{\ell m}$ denotes a spherical harmonic.)

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